On Some Scientific Results of Abid-Waheed Graph $(AW)_r^7$

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Abstract

Graphs are the representation of data in pictorial form. The use of graph theory is increasing day by day. Abid Waheed graph can be utilized in computer networking. The graph invariant or topological index is a numeric number used to explore the properties of the different graphs without experiments just by correlation coefficients. The Wiener index and its modification called hyper wiener index are both distance-based. In this paper we calculate the Wiener index (WI), hyper Wiener (HW) index of newly defined ”Abid Waheed graph $(AW)_r^7$ by the mean of Hosoyapolynomial.

Keywords: Graph theory, Topological index, Distance, Hosoya polynomial, Wiener index, Abid Waheed Graph.

1. Introduction

Graphs are mathematical representations of networks that show the relationship between points and lines of the network. Leonhard Euler [1], one of the most renowned mathematicians of the 18th century developed the idea of graphs in the 18th century. Graph theory is commonly associated with his work on the famous ”Seven Bridges of Knigsberg” problem. In physical, biological, social, and information systems, graphs can be used to model many types of relationships and processes. In recent days graph theory has been useful in finding communities in networks, including social media, and identified COVID19 in the community through contacts, determining rankings for web links in search engines, identifying molecules and atoms in chemistry, and studying computer network security [2-5].

The mathematics of chemistry is the subject of research concerned with new applications of mathematics. A molecular graph in mathematical chemistry can be characterized by a variety of topological indices, and these are invariant with respect to graph automorphism. The topological index is a numerical quantity that is derived from the structure graph of a molecule in an unambiguous manner. For more investigation about TIs see [6-9]. The graph invariant, no matter how graph is labeled or illustrated doesn’t depend on its rep- resentation. A number of topological indices can explain molecular size and shape.

Hosoya polynomials are a generating

$$W(\breve{G}) = \frac{1}{2} \sum_{\alpha,\beta \subset E(\breve{G})} \Upsilon(\alpha,\beta)$$

function created by Haruo Hosoya [10] in 1988 for distance distribution for graphs. The hyper wiener index can be calculated

$$H(\breve{G},u) = \sum_{\alpha,\beta \in E(\breve{G})} \Upsilon(\alpha,\beta) u^{\Upsilon(\alpha,\beta)}$$

by different methods such as the interpolation method, cut method and method of Hosoya polynomials.

A large number of papers have been published on the Wiener index because of its useful applications. A very famous mathematician Gutman deeply study this index for the tree graphs in 2001. He expose the tree graphs in chemistry and mathematics with the help of this index [11]. As we are well familiar with the use of indices in chemistry but Tepeh apply this index in mathematics in 2015 [12]. There are many approaches defined for the W-index as vertex approach, edges approach. The edge version of W-index is defined by Gutman and Iranmanesh in 2009 [13]. To understand the concept of terminal W-index see [14]. Gong and

Hua introduce this index in physics for the

$$W(\breve{G}) = \frac{\partial H(\breve{G},u)}{\partial u}\Big|_{u=1}$$

bi-polar fuzzy graphs [15]. Different version of Wiener index is studied by Hammer and Donno for the Bassica graph [16].

2. Preliminaries

Let Gˇ be a connected and simple graph with E(Gˇ) is the collection of edges and V (Gˇ) is the collection of nodes. The degree of the vertex α is the quantity of edges connected with the vertex α denoted by Y(α) or (α, β). An edge is formed by the connection of two different vertices α and β representing as αβ. The graph in this paper is newly defined and has resemblance with Jahangir graph.

An index of Harry Wiener is calculated by adding the distances between all vertices of a graph Gˇ. The first index known as wiener index mathematically described as:

In the so-called inverse structure property relationship problems, the Wiener index plays a crucial role. A Japanese chemist gave a concept parallel to the TIs called Hosoya polynomial in 1988.

Where Y(α, β) shows the distance between α and β. Hosoya polynomials grasp many metric properties of a graph, for example, the W-index (average distance) and WW-index. Raza Farhani study the Jahangir graph with the help of W-index and H-polynomial [17]. To understand the variation of H-polynomial on the wheel and den graph check out [18].

The first time differentiation of H-polynomial at u=1 give us the W-index. To find the W-index with the help H-polynomial is very short and easy approach. The W-index by the mean of H-polynomial is defined as:

The modified form of W-index is hyper wiener index was introduced by Milan Randic in 1993 [25]. It is a distance based TI calculated by inspiring the progress of W-index. In some fields like networking, WW- index is more effective to tell the properties of network by distance. It can also calculated by the help of H-polynomial. Its formula involves first and second derivatives of H- polynomial as:

$$WW(\check{G}) = H'(\check{G},u) + \frac{1}{2}H''(\check{G},u)$$

$$D(\check{G}) = \max_{\alpha \in V(\check{G})} \Upsilon(\alpha,\beta) \text{ for all } \alpha \in V(\check{G})$$

The general formula of WW-index is:

$$WW(\check{G}) = \frac{1}{2}(\sum_{\alpha\beta\varepsilon E(\check{G})} \Upsilon($$

Recently Peng and Wakar Ahmed deeply study the relationship between these W-index and WW- index for polygonal Cylinder and Torus [20]. WW-index of the fuzzy graph has applications in the share market. So it is beneficial for the marketing graphs. In 2021 a paper was published named by Wiener type indices describe these indices for the Parikh word respectable graph [22]. There are various methods to find the hyper Wiener index, to understand these methods see [23]. As Randic proposed the WW-index for the acyclic graphs but it is also defined for the cyclic graphs.

The longest topological distance in a graph is called topological diameter $D(\check{G})$ mathematically stated as:-

Composite graph is also an important graph in mathematics. The H-polynomial of this graph is computed by Stevanovic [19].

3. Construction of Abid Waheed $(AW)^s_r$ graph for $s \geq 3$ and $r \geq 1$

In this section, we describe the construction and methodology for our new notion of a connected graph known as Abid Waheed (AW-graph) an defined as;

Abid Waheed graph $(AW)^s_r$ consists of rs+1 vertices and r(s+1) edges for all r 1 and s graph generated by r-cycles (with each of order s), meeting at an external vertex of degree r.It is denoted by $(AW)^s_r$ for s ≥ 3 and r ≥ 1. Representation of Abid-Waheed Graph Shown in Figure 1.

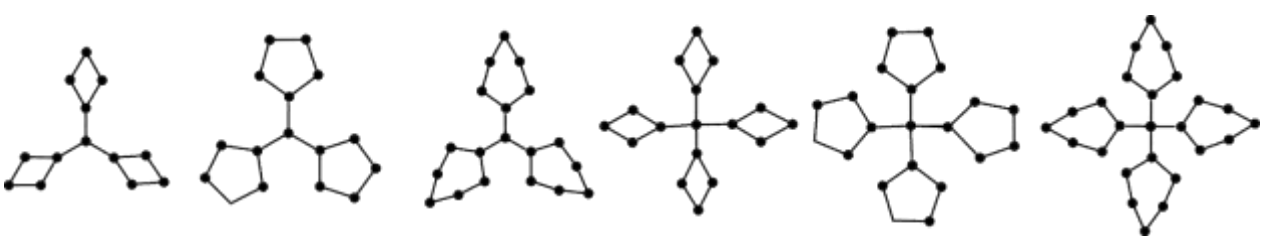

Figure 1: Abid-Waheed Graph $(AW)^s_r$

4. Fundamental Results

In this section, we try to investigate the characteristics of Abid Waheed graph with the help of TIs such as Wiener index WI and hyper wiener index WWI. The hyper W-index is just modification of path number (wiener index). These indices can be calculated by derived formulae but we use the polynormial approach because polynomial method is very easy and short to find the indices. The basic purpose of this section is to study the Abid Waheed graph $(AW)_r^7$ for $r \geq 1$. We get W-index with the help of differentiation of Hsosya polynomial at u=1.

Example

In this example we are intersted to discuss the H-polynomial, W-index and WW-index of a single abid Waheed graph $(AW)_3^7$. The values and calculations of $(AW)_3^7$ are given below:-

$$\Rightarrow H(\check{G}, u) = \sum_{p=1}^{8} \Upsilon(\check{G}, p)u^p$$

In Figure 1 (a), the Abid Waheed graph for r=3 and s=7 is presented. The highest distance between the edges $(AW)_3^7$ is 8, that means no path between the edges of has length greater than eight. So the T-diameter of this graph is 8 as given in Table 1. All the eight distances with the respective cardinalities are calculated in Table 1. With the help of values given in table we can easy compute the H-polynomial and then by utilizing this polynomial we will calculate pour required indices.

$\Upsilon(\check{G}, p)$	Cardinality	$\Upsilon(\check{G}, p)$	Cardinality
$\Upsilon(\check{G}, 1)$	24	$\Upsilon(\check{G}, 5)$	36
$\Upsilon(\check{G}, 2)$	30	$\Upsilon(\check{G}, 6)$	36
$\Upsilon(\check{G}, 3)$	39	$\Upsilon(\check{G}, 7)$	24
$\Upsilon(\check{G}, 4)$	30	$\Upsilon(\check{G}, 8)$	12

Table 1: Distance $\Upsilon(\check{G}, p)$ for $1 \leq p \leq 8$ for every pair of nodes

$$H(\check{G}, u) = \Upsilon(\check{G}, 1)u + \Upsilon(\check{G}, 2)u^2 + \Upsilon(\check{G}, 3)u^3 + \Upsilon(\check{G}, 4)u^4 + \Upsilon(\check{G}, 5)u^5 + \Upsilon(\check{G}, 6)u^6 + \Upsilon(\check{G}, 7)u^7 + \Upsilon(\check{G}, 8)u^8$$

Putting the values of distances from Table 1, we get:

$$H((AW)_3^7, u) = 24u + 30u^2 + 39u^3 + 30u^4 + 36u^5 + 36u^6 + 24u^7 + 12u^8$$

By differentiating the above equation only one time at u=1 give us the W-index. So the W-index is determined as:-.

$W((AW)_3^7) = 24 + 60u + 117u^2 + 120u^3 + 180u^4 + 216u^5 + 168u^6 + 96u^7$

$W((AW)_3^7) = 981$

The WW-index for $(AW)_3^7$ can be determined by using the formula but we use H-polynomial to calculate its values:-

$$WW(\breve{G}) = H'((AW)_3^5, u) + \frac{1}{2}H''((AW)_3^7, u)$$

As we already have the 1^{st} derivative of H-polynomial:

$$H^t((AW)_3^7, u)|_{u=1} = 981$$

The 2nd time differentiation of polynomial and its values at u=1 are:

$$H''((AW)_3^7, u) = 60 + 234u + 360u^2 + 720u^3 + 1080u^4 + 1008u^5 + 672u^6$$
$$H''((AW)_3^7, u)|_{u=1} = 4134$$

By substituting the values in fomula, we get WW-index for $(AW)_3^7$

$$WW((A\breve{W})_3^7) = 981 + \frac{4134}{2}$$
$$WW((A\breve{W})_3^7) = 3048$$

Example

The W-index involve only one derivative of H-polynomial but WW-index is just the next step of this index because we deal with both first and second derivatives of H-polynomial. The W-index and WW-index are determined below by computing the H-polynomial for the Abid WAheed graph $(AW)_4^5$

$$H(\breve{G}, u) = \sum_{p=1}^{8} \Upsilon(\breve{G}, p)u^p$$

The 2D graph of $(AW)_4^7$ is presented in Figure 1(b) where s=7 and r=4. The maximum distance that is called T-diameter of $(AW)_4^7$ is eight. We have counted all the 8 distances with the definition of Abid Waheed graph and H-polynomial. These all distances from 1 to 8 are given in Table 2 with the all frequencies. All the information related to both indices given in table.

$\Upsilon(\breve{G},p)$	Cardinality	$\Upsilon(\breve{G},p)$	Cardinality
$\Upsilon(\breve{G},1)$	32	$\Upsilon(\breve{G},5)$	72
$\Upsilon(\breve{G},2)$	42	$\Upsilon(\breve{G},6)$	72
$\Upsilon(\breve{G},3)$	60	$\Upsilon(\breve{G},7)$	48
$\Upsilon(\breve{G},4)$	56	$\Upsilon(\breve{G},8)$	24

Table 2: Distance $\Upsilon(\breve{G}, p)$ for $1 \le p \le 8$ for every pair of nodes

$$H(\breve{G}, p) = \Upsilon(\breve{G}, 1)u + \Upsilon(\breve{G}, 2)u^2 + \Upsilon(\breve{G}, 3)u^3 + \Upsilon(\breve{G}, 4)u^4 + \Upsilon(\breve{G}, 5)u^5 + \Upsilon(\breve{G}, 6)u^6 + \Upsilon(\breve{G}, 7)u^7 + (\breve{G}, 8)u^8$$

On the base of Table 2 we have H-polynomial of $(AW)^7_4$:

$$H((AW)^7_4, u) = 32u + 42u^2 + 60u^3 + 56u^4 + 72u^5 + 72u^6 + 48u^7 + 24u^8$$

We compute the W-index of $(AW)^7_4$ with the help of H-polynomial by taking the differentiation at u=1 as:-

$$W(\breve{G}) = 24 + 84u + 180u^2 + 224u^3 + 360u^4 + 432u^5 + 336u^6 + 192u^7$$

$$W(\breve{G})|_{u=1} = 1840$$

The WW-index for $(AW)^7_4$ can be determined by formula as well as H-polynomial. We prefer the polynomial method to explore WW-index as:-

$$WW(\breve{G}) = H'((AW)^7_4, u) + \frac{1}{2}H''((AW)^7_4, u)$$

We already fined out the exact values of W-index that is basically the 1st order differentiation of H-polynomial.

$$H'(\breve{G}, u)|_{u=1} = 1840$$

And the second order derivatives of H-polynomial with the values at u=1 are computed as:

$$H''(\breve{G}, u) = 84 + 360u + 672u^2 + 1440u^3 + 2160u^4 + 2016u^5 + 1344u^6$$
$$H''(\breve{G}, u) = 8076$$

Now putting the values of both derivatives in the formula of WW-index we have:-

$$WW(\breve{G}) = 1840 + \frac{8076}{2}$$
$$WW(\breve{G}) = 5878$$

The basic purpose of this section is to find the generalized form of W-index, H-polynomial and WW-index for special class of Abid Waheed graph $(AW)^7_r$ for r 1. We can generalized the calculations by hand or use Matlab software that is time saving, exact and error free method.

Theorem 4.1. *Consider Abid Waheed graph $\check{G} \cong (AW)^7_r$ for $r \geq 1$ where r is an integer then the H-polynomial of $(AW)^7_r$ is:-*

$$H(\check{G},p) = 8ru + (\frac{r^2+17r}{2})u^2 + (2r^2+7r)u^3 + (4r^2-2r)u^4 + (6r^2-6r)u^5 + (6r^2-6r)u^6$$

Proof. Suppose $(AW)^7_r$ is Abid Waheed graph for r 1. The count of edges is 8r and the total number of nodes are 7r+1. The Abid Waheed graph $(AW)^7_r$ has total 8 types of distances. The maximum distance is 8 that is called T-diameter in simple words. These all different distances and their cardinalities are given in the Table 3.

$\Upsilon(\check{G},p)$	Cardinality	$\Upsilon(\check{G},p)$	Cardinality
$\Upsilon(\check{G},1)$	8r	$\Upsilon(\check{G},5)$	$6r^2$-6r
$\Upsilon(\check{G},2)$	$\frac{r^2+17r}{2}$	$\Upsilon(\check{G},6)$	$6r^2$-6r
$\Upsilon(\check{G},3)$	$2r^2$+7r	$\Upsilon(\check{G},7)$	$4r^2$-4r
$\Upsilon(\check{G},4)$	$4r^2$-2r	$\Upsilon(\check{G},8)$	$2r^2$-2r

Table 3: Distance $\Upsilon(\check{G},p)$ $1 \leq p \leq 8$ for every pair of nodes $(AW)^7_r$

The H-polynomial of $(AW)^7_r$ can be determined on the base of Table 3:

$$H(\check{G},u) = \sum_{p=1}^{D(\check{G})} \Upsilon(\check{G},p)u^p$$

As we have the total eight distances so:

$$H(\check{G},u) = \Upsilon(\check{G},1)u + \Upsilon(\check{G},2)u^2 + \Upsilon(\check{G},3)u^3 + \Upsilon(\check{G},4)u^4 + \Upsilon(\check{G},5)u^5 + \Upsilon(\check{G},6)u^6 + \Upsilon(\check{G},7)u^7 + \Upsilon(\check{G},8)u^8$$

By utilizing Table 3 we have:

$$H((AW)^7_r,p) = 8ru + (\frac{r^2+17r}{2})u^2 + (2r^2+7r)u^3 + (4r^2-2r)u^4 + (6r^2-6r)u^5 + (6r^2-6r)u^6 + (4r^2-4r)u^7 + (2r^2-2r)u^8$$

Theorem 4.2. *Consider the graph $\check{G} \simeq (AW)^7_r$ for $r \geq 1$, then W-index of $(AW)^7_r$ will be:-*

$$W(\check{G}) = 133r^2 - 72r$$

Proof. Consider the Abid Waheed $(AW)_r^7$ where r ≥ 1. On the base of mathematical definition of H-polynomial and W-index we can easily compute the general form of W-index for AW-graph:

$$W(\check{G}) = \frac{\partial H(\check{G}, u)}{\partial u}|_{u=1}$$

$$W(\check{G}) = 8r + (\frac{r^2 + 17r}{2})2u|_{u=1} + (2r^2 + 7r)(3u^2)|_{u=1} + (4r^2 - 2r)(4u^3)|_{u=1} + (6r^2 - 6r)(5u^4)|_{u=1} + (6r^2 - 6r)(6u^5)|_{u=1} + (4r^2 - 4r)(7u^6)|_{u=1} + (2r^2 - 2r)(8u^7)|_{u=1}$$

After some easy steps of calculations we have:

$$= 133r^2 - 72r$$

The WW-index for the $(AW)_r^7$ is determined as by the H-polynomial approach. This is short, efficient and time saving method for the WW-index. The simple formula is also present in the basic definitions to determine the WW-index.

Theorem 4.3. *Consider Abid Waheed graph $\check{G} \simeq (AW)^7$ for all $r \geq 1$ then its WW-index is:*

$$WW(\check{G}) = \frac{907}{2}r^2 - \frac{689}{2}r$$

Proof. Let Abid Waheed $(AW)_r^7$ with $r \geq 1$. To find the WW-index we must have proper understanding with the W-index and H-polynomial. Just using these two terms and definition of WW-index we will be able to find out the exact values of WW-index.

$$W(\check{G}) = H^t(\check{G}, u)|_{u=1}$$

We have find the values of W-index in Theorem 2.

$$W((AW)_r^7) = 133r^2 - 72r$$

The other term in the definition of WW-index is the 2nd order derivative of H-polynomial, so that 2nd order derivative is:

$$H''(\check{G}, u) = \frac{\partial H'(\check{G}, u)}{\partial u}$$

Putting the values from Table 3:

$$H''(\check{G}, u) = (r^2 + 17) + (12r^2 + 42r)u + (48r^2 - 24r)u^2 + (120r^2 - 120r)u^3 + (180r^2 - 180r)u^4 + (168r^2 - 168r)u^5 + (112r^2 - 112r)u^6$$

$$H''(\check{G}, u)|_{u=1} = 641r^2 - 545r$$

By utilizing formula of WW-index given in preliminaries and by substituting values we get:

$$WW(\breve{G}) = (133r^2 - 72r) + \frac{1}{2}(641r^2 - 545r)$$

$$WW(\breve{G}) = \frac{907}{2}r^2 - \frac{689}{2}r$$

5. Conclusion

Graph theory is an area of mathematics and computer science that deals with graphs, or diagrams containing points and lines and which represent mathematical truths pictorially. It has a broad scope of applications. The use of graph theory has exponentially increased. It is effective to understand flow of computation, networks of communication, data organization and Google maps in computer. In this article, we calculate some very important results related to Abid- Waheed graph. Abid-Waheed graphs have great importance in electrical engineering (design electrical connections), Linguistics (parsing of language tree, grammar of a language tree, phonology, and morphology), chemistry, physics, mathematics and biology. In future, we can describe applications of Abid-Waheed graph in computer networking.

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This research received no founding.

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